Relativity, Thermodynamics and Entropic Forces

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A recent assertion that inertial and gravitational forces are entropic forces due to changes in the information associated with the positions of bodies is considered. Herein, the entropic force is shown to be a natural consequence of conventional thermodynamics and relativity, without appealing to the information proposal. An interpretation of the temperature and entropy of an accelerating body is then developed to link the entropic force to Newton's second law of motion. The entropic force experienced by an observer residing in general coordinates is also derived and then expressed in Schwarzschild coordinates, leading to an expression of the entropy change of in-falling matter due to gravitation. As a final consideration, the total entropy associated with an ordinary, Newtonian gravitational field is derived, demonstrating that as in-falling matter loses entropy, the gravitational field gains entropy. In closing, it is proposed that the entropy of a gravitational field is proportional to the total stress-energy-momentum associated with the matter comprising the gravitational source.

I. INTRODUCTION

One of the longest standing questions in physics certainly has been by what means does ordinary matter resist acceleration (inertia). A traditional approach to this problem has been to view inertia as merely a fundamental property of all matter with no further explanation attainable. Throughout the years, however, many have sought to explain inertia by expressing the inertial mass m appearing in Newton's second law of motion,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{1}$$

in terms of some other more fundamental entity or interaction. One approach, put forth within the Standard Model of particle physics, proposes a background scalar Higgs field, pervading all of space, by which otherwise massless elementary particles acquire their mass. Another approach proposes that the inertial properties of ordinary matter are due to scattering of a quantum vacuum electromagnetic zero-point field (ZPF) by subatomic particles constituting ordinary matter. According to the ZPF proposal, this scattering of radiation exerts a reactive body force on the subatomic particles, which can be associated with the inertial mass of matter.[1–3]

A more recent proposal asserts that gravitational and inertial forces are entropic forces caused by changes in the information associated with the positions of material bodies. According to the information proposal, gravitational and inertial forces both originate from one general phenomenon, expressed as an entropic force, given by[4]

$$\mathbf{F} = T\nabla S \tag{2}$$

in which T is temperature, and S is entropy. A conceptual difficulty, however, is that in using (2), one seemingly must define T and S so as to retrieve (1) in the case of acceleration, and $\mathbf{F} = -m\nabla\phi$ in the case of gravitation. Rather than merely selecting functions for T and S as appears to be done in the literature,[4] herein the entropic force is shown to be a natural consequence of conventional thermodynamics and relativity theory, without appealing to the information proposal.

In the next section, a material body undergoing uniform acceleration in Minkowski space-time is considered. The body is treated as a closed thermodynamic system, which is subjected to an external force by a heat engine carried by a nearby stationary observer. In applying the first law of thermodynamics to the moving body, the internal energy of body is assumed to remain unchanged, and thus the acceleration is treated as an isothermal process. Moreover, it is pointed out that since the force applied by the heat engine does positive work on the body, the work performed by the body is negative. An expression of the above-mentioned entropic force results upon noting that the work performed by the body equals the negative change in potential energy of the body, taken in the limit of infinitesimal displacement. The resulting expression of the entropic force is identical to (2), with the exception of a minus sign. It is pointed out, however, that the minus sign suggests that the entropy of the body decreases during the acceleration, while the entropy of the heat engine increases as required by the second law of thermodynamics. As a final consideration, an expression for the entropy of the accelerating body is developed, thereby reducing the entropic force to Newton's second law of motion, (1).

Section III is devoted to deriving the entropic force experienced by an observer residing in general coordinates. The section begins with a derivation of the change in entropy of a test body based on the change in potential energy of the body at two locations in the coordinate system. An expression of the entropic force in general coordinates is then obtained in the limit of infinitesimal displacement. Next, the change in entropy of the test body is expressed in Schwarzschild coordinates. The resulting expression indicates that the entropy of in-falling matter decreases as the gravitational field performs positive work. The section closes with a proposal that the second law of thermodynamics requires that the entropy of the gravitational field increases as matter is drawn in and captured.

In section IV, the expression of the gravitational entropy derived in section III is used to demonstrate that the entropy of a Newtonian gravitational field is proportional to the total self-energy of the gravitational source. This is carried out by using an approach analogous to that used to derive the self-energy of a charge distribution in electrostatics. Specifically, pairs of particles of matter are envisioned to gravitate toward one another so as to build up an idealistic gravitational source body. As the number of particles increases, so too does the total positive entropy associated with the resulting gravitational field. It is then proposed that the total energy associated with the gravitational source, including any internal stress and thermodynamic phenomena that may be present, should contribute to the gravitational entropy, as well. The resulting expression indicates that as infalling matter loses entropy, the gravitational field gains entropy as required by the second law of thermodynamics. As the size of the gravitational source grows, so too does its entropy. It is pointed out that were the source the size of a star which later collapsed into a black hole, the gravitational entropy would then be associated with the event horizon of the black hole.

II. ENTROPIC INERTIAL FORCE

As mentioned in the introduction, a recent information-based proposal asserts that gravitational and inertial forces are entropic forces caused by changes in the information associated with the positions of material bodies.[4] As shown below, however, adhering to conventional thermodynamic concepts leads naturally to the entropic force given by (2), as well as providing straightforward definitions of T and S, leading to (1).

Let a material body be a closed thermodynamic system situated in Minkowski space-time. Nearby is a stationary observer that uses a heat engine to apply an external force to the body such that the state of the body goes from state A to state B while the body accelerates uniformly over a distance Δx. The change in entropy of the body can be determined by using the familiar expression

$$S(B) - S(A) = \int_A^B \frac{dQ}{T} \tag{3}$$

where Q is heat energy, and T is temperature. According to the first law of thermodynamics, the heat energy dQ entering or leaving the body is given by

$$dQ = dE + dW \tag{4}$$

where dE is the change in internal energy, and dW is the work performed on or by the body. Assuming the internal energy of the body remains unchanged during the acceleration, we may hold T constant and put $dE = 0$. Moreover, since the force applied by the heat engine does positive work on the body, the work performed by the body is negative. Under these conditions, (4) reduces to

$$dQ = -dW. \tag{5}$$

Using (5) in (3) and noticing that the change in potential energy of the body can be expressed as $\Delta U = -\Delta W$ puts (3) in the form

$$\Delta U = T\Delta S. \tag{6}$$

As pointed out above, the change in entropy ΔS occurs over the distance Δx. With this in mind, let us take the limit of (6) as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = T \lim_{\Delta x \to 0} \frac{\Delta S}{\Delta x}. \tag{7}$$

Carrying out the limit on both sides of (7) gives

$$\nabla U = T\nabla S. \tag{8}$$

Upon noting that in general the force on the body is $\mathbf{F} = -\nabla U$, (8) can then be recast in the form

$$\mathbf{F} = -T\nabla S. \tag{9}$$

Equation (9) is equivalent to the entropic force given by (2), with the exception of a minus sign.

At this point, it is worthwhile to mention that the minus sign makes good thermodynamic sense. As the heat engine accelerates the body, the entropy of the body decreases as the subatomic particles comprising the body are put into a state increasingly capable of performing positive work. At the same time, the entropy of the heat engine increases, which we may denote by $\Delta S_{he} > 0$. It is conceptually straightforward to see that the total entropy of the heat engine and the accelerating body increases as the body accelerates, according to $\Delta(S_{he} + S) \geq 0$ as required by the second law of thermodynamics.

An expression equivalent to (9) can be derived by noticing that the total energy of the moving body, as viewed by the stationary observer, can be expressed in general as

$$U = U_0 \frac{dt}{d\tau} \tag{10}$$

in which dt is an interval of time for the stationary observer, and $d\tau$ is an interval of proper time for the accelerating body. Substituting (10) into $\mathbf{F} = -\nabla U$ then gives

$$\mathbf{F} = -U_0 \nabla \left(\frac{dt}{d\tau} \right). \tag{11}$$

It will be recognized that (11) is just the inertial reaction force of the moving body as viewed by the stationary observer in Minkowski space-time.[5] The equivalence of (9) and (11) can be used to express the temperature T and entropy S in a form that relates (9) to Newton's second law of motion, (1). It will be recalled that T was held constant in the derivation of (9). To determine S, however, let us equate (9) and (11) and then solve for the entropy change ΔS. Carrying this out leads to

$$\Delta S = \frac{U_0}{T}\Delta\gamma \tag{12}$$

where γ is used as a convenient notation for $dt/d\tau$. Assuming that the body is initially at rest before the force is applied, (12) can be expressed as

$$\Delta S = \frac{U_0}{T}\left(\gamma - 1\right). \tag{13}$$

Equation (13) can be applied specifically to the accelerating body by noting that for weak, Newtonian acceleration, we may put[6,7]

$$\gamma \approx 1 - \frac{a\Delta x}{c^2} \tag{14}$$

where a is the proper acceleration of the body, and Δx is the distance in flat space-time over which the acceleration takes place. Substituting (14) into (13) and using the expression $U_0 = m_0 c^2$ leads to

$$\Delta S \approx -\frac{m_0 a}{T}\Delta x \tag{15}$$

where it is to be understood that the mass m_0 represents the total inertial mass due to all forms of stress and energy associated with the accelerating body. Equation (15) used in conjunction with (9) leads directly to Newton's second law of motion, (1). Equation (15) also indicates that for any given temperature, the entropy of the accelerating body decreases as positive work is performed on the body. As mentioned above, however, the entropy of the stationary heat engine increases as it performs positive work to accelerate the body, $\Delta S_{he} > 0$. As required by the second law of thermodynamics, the net effect of applying the force to the body is an increase in the entropy of the universe, given by $\Delta(S_{he} + S) \geq 0$.

III. GENERAL COORDINATES

In the previous section, the force on a moving body was considered from the vantage of a stationary observer is Minkowski space-time. Now, let us consider the entropic force on a test body according to an observer residing in an enclosed vessel stationed in general coordinates.

Suppose the observer measures the weight of the test body at one wall of the vessel, say wall A, and then carries out the same measurement at the opposite wall of the vessel, called wall B. According to the observer, the state of the test body changes from the position at wall A to wall B. Upon determining the change in potential energy of the body, the observer can use (6) to determine the change in entropy of the test body occurring between these two states:

$$\Delta S = \frac{1}{T}\left(U_B - U_A\right) \tag{16}$$

where U_A and U_B are the total potential energy of the test body at walls A and B, respectively, and the temperature T of the test body is assumed to remain constant. Suppose the energy of the test body at wall A is

$$U_A = U_0 \tag{17}$$

where U_0 is the total energy associated with the body according to the observer in the vessel. Relative to wall A, the energy of the body at wall B can put in the form

$$U_B = U_0\frac{d\tau_B}{d\tau_A} \tag{18}$$

where $d\tau_A$ and $d\tau_B$ are intervals of proper time experienced at walls A and B, respectively. Substituting (17) and (18) into (16) puts the entropy change of the test body in the form

$$\Delta S = \frac{U_0}{T}\left(\frac{d\tau_B}{d\tau_A} - 1\right). \tag{19}$$

Upon expressing things in terms of an interval of coordinate time dt, (19) assumes the form

$$\Delta S = \frac{U_0}{T}\left(\frac{dt}{d\tau_A}\right)\left(\frac{d\tau_B}{dt} - \frac{d\tau_A}{dt}\right) \tag{20}$$

Let us suppose that the path of travel of the test body in moving from state A to state B is aligned with the $i-$coordinate direction. The directional change in entropy with respect to Δx^i is then expressible as

$$\frac{\Delta S}{\Delta x^i}\mathbf{g}_i = \frac{U_0}{T}\left(\frac{dt}{d\tau_A}\right)\left(\frac{d\tau_B/dt - d\tau_A/dt}{\Delta x^i}\right)\mathbf{g}_i \tag{21}$$

where $\mathbf{g}_i$ is a general basis vector pointing in the $i-$coordinate direction. Taking the limit of (21) as $(d\tau_B, d\tau_A) \to d\tau$ and $\Delta x^i \to 0$, and rearranging a bit leads to

$$-T\nabla S = U_0\gamma^{-1}\nabla\gamma \tag{22}$$

where $\gamma = dt/d\tau$ has been used for convenience. It will be recognized that the right-hand side of (22) is just the force on the test body due to the general coordinate system.[5,8] Thus, the entropic force experienced by the observer holding the body stationary is simply

$$\mathbf{F} = -T\nabla S \tag{23}$$

which at first sight appears to be identical to (9), derived in flat space-time. It must be kept in mind, however, that the gradient operator in general coordinates assumes the form[5,9]

$$\nabla \to -\mathbf{g}_i g^{ij}\partial_j \tag{24}$$

where $\mathbf{g}_i$ is the above-mentioned general basis vector pointing in the $i-$coordinate direction, and the minus sign is included due to our choice of sign convention,

$(+,-,-,-)$. Using (24) puts the entropic force in the general form

$$\mathbf{F} = T\mathbf{g}_i g^{ij} \partial_j S \tag{25}$$

Now let us consider the change in entropy of the test body due to gravitation near a large, spherically symmetric source of mass M. This is most easily carried out by expressing (22) in Schwarzschild coordinates. The redshift factor γ in Schwarzschild coordinates is $\gamma = (1+2\phi/c^2)^{-1/2}$, where ϕ is the field potential outside the gravitational source.[8] Using (24) and the Schwarzschild γ in (22) leads to

$$\Delta S_m = \frac{m}{T}\gamma^2 \Delta\phi \tag{26}$$

where the subscript on S_m denotes the entropy associated with the test body. Thus, the entropy of the test body scales proportionally to the gravitational field potential ϕ. Noting that the field potential ϕ is negative relative to distant, star-fixed observers, (26) indicates that if the test body is allowed to fall, the entropy of the body decreases as the body approaches the gravitational source. Unlike the case of uniform acceleration considered in the previous section, here there is no heat engine applying an external force on the test body. Rather, the motion of the body is due to space-time curvature near the gravitational source. Of course, upon approaching this problem from a Newtonian standpoint, the gravitational field can be viewed as performing positive work on the observer. With the second law of thermodynamics in mind, it seems logical that while the gravitational field performs positive work on the test body, the entropy of the gravitational field increases according to

$$\Delta S_g \geq -\frac{m}{T}\gamma^2 \Delta\phi \tag{27}$$

where the subscript on S_g denotes the entropy associated with the gravitational field, and it should be understood that the minus sign is included in (27) simply because ϕ is negative relative to distant observers, as mentioned above. Combining (26) and (27) suggests that the total entropy change of the universe due to the test body gravitating toward the gravitational source is $\Delta\left(S_g + S_m\right) \geq 0$.

IV. ENTROPY OF MATTER

Entropy is well known to be associated with black holes.[10–12] Perhaps, not as well known is that entropy can also be associated with ordinary, Newtonian gravitational bodies. The entropy associated with Newtonian gravitation can be derived by using an approach analogous to that used to derive the self-energy of a charge distribution in electrostatics. Specifically, pairs of constituent particles of matter are envisioned to gravitate toward one another so as to build up an idealistic gravitational source body. As the number of particles increases, so too does the size of the body and the total entropy of the resulting gravitational field. With this approach in mind, let us recast (27) in the Newtonian limit for the case of $N-$many gravitating particles as

$$S_g \geq -\frac{1}{2T}\sum_{i=1}^{N} m_i \sum_{\substack{j=1 \\ j\neq i}}^{N} \phi_j \tag{28}$$

in which the factor of 1/2 is included so as to account for duplicate pairs of particles in the sum. Equation (28) can be simplified by noticing that the right-most sum is just the total field potential $\phi\left(P_i\right)$ at the position P_i of m_i. Thus, (28) can be expressed simply as

$$S_g \geq -\frac{1}{2T}\sum_{i=1}^{N} m_i \phi\left(P_i\right). \tag{29}$$

Upon generalizing to the case of a continuous volume distribution of matter, (29) can be recast in the form

$$S_g\left(\mathbf{x}\right) \geq -\frac{1}{2T}\int \rho\left(\mathbf{x}'\right)\phi\left(\mathbf{x}'\right) d^3x' \tag{30}$$

where $\rho\left(\mathbf{x}'\right)$ is the mass-density of the matter distribution, $\phi\left(\mathbf{x}'\right)$ is the gravitational potential, and primed quantities are taken over the volume of the matter distribution. As the reader will recall, $\mathbf{x}$ points from the origin to a position from which observations are performed, and $\mathbf{x}'$ points from the origin to a position occupied by an elementary portion of the matter distribution.[13]

Upon expressing (30) in terms of the gravitational self-energy of a continuous matter distribution, given by

$$U_s = -\frac{1}{2}\int \rho\left(\mathbf{x}'\right)\phi\left(\mathbf{x}'\right) d^3x' \tag{31}$$

the entropy of the gravitational field assumes the very simple form

$$S_g \geq \frac{U_s}{T}. \tag{32}$$

Equation (32) suggests that the total entropy of the Newtonian gravitational field is proportional to the total self-energy of the gravitational source. Equation (32) taken in view of (26) indicates that while in-falling matter loses entropy, the entropy of the gravitational field increases. As a further generalization of (32), it makes sense to include contributions due to the stress-energy-momentum of the constituent particles, as well. This suggests putting $U_s = Mc^2$, in which M is the total gravitational mass due to all sources of energy associated with the body, including internal stresses and any thermodynamic phenomena that may be present, as well.[14–16] Based on this line of reasoning, (32) then assumes the form

$$S_g \geq \frac{Mc^2}{T}. \tag{33}$$

Equation (33) agrees in principle with the expression $T^{-1} = \left(\partial S/\partial Mc^2\right)$ familiar to black hole mechanics.[11] Although the body considered here is not a black hole, and thus has no event horizon, it should be understood that while M and T are the mass and temperature of the source body, the left-hand side of (33) is the entropy associated with the gravitational field of the body, not the source body itself. Were this idealistic body allowed to assume the characteristics of a star that later collapsed into a black hole, the entropy given by (33) would then be associated with the event horizon of the black hole, as is well known in the literature.[10–12]

V. DISCUSSION AND CONCLUSIONS

As discussed in the introduction, a recent proposal asserts that gravitational and inertial forces are entropic forces caused by changes in the information associated with the positions of material bodies.[4] As pointed out, however, a drawback to the information proposal is that temperature and entropy must be defined properly so that the entropic force leads to known expressions of the inertial and gravitational forces. Herein, the entropic force was derived on the basis of conventional thermodynamics and relativity without using the information-based approach.

In section II, the first and second laws of thermodynamics were applied to the case of a body undergoing uniform acceleration due to an external force applied by a heat engine. This conventional approach led directly to an expression of the entropic force identical to that known in the literature,[4] with the exception of a minus sign. Upon associating the entropic force with Newton's second law of motion, the minus sign made it clear that the entropy of the accelerating body decreases as positive work is performed on the body by the heat engine. At the same time, the entropy of the heat engine increases. It was concluded that the net effect of applying the force to accelerate the body is an increase in the entropy of the universe.

Section III was devoted to deriving the entropic force experienced by an observer residing in general coordinates. This was carried out by determining the difference in potential energy of a test body at two locations in the coordinate system and then calculating the corresponding entropy change of the body. Taken in the limit of infinitesimal displacement, the entropy change led to an expression of the entropic force in general coordinates. The entropy change was then expressed in Schwarzschild coordinates, at which point it became clear that the entropy of in-falling matter decreases as the gravitational field performs positive work. The section closed with a proposal that the second law of thermodynamics requires the entropy of the gravitational field to increase as matter is drawn in and captured.

In section IV, the expression of the gravitational entropy derived in section III was used in an idealistic derivation which demonstrated that the entropy of a Newtonian gravitational field is proportional to the total self-energy of the gravitational source. It was then suggested that the total stress-energy-momentum associated with the source ought to contribute to the entropy, as well. The resulting expression indicates that as in-falling matter loses entropy, the gravitational field gains entropy. As the size of the gravitational source grows, so too does its entropy. It was pointed out that were this idealistic body a black hole, the entropy would be associated with the event horizon of the black hole.[10–12]

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