Gravitational Force near a Rotating Black Hole

Charles T. Ridgely*
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Rarely is the gravitational force on matter computed explicity in textbooks dealing with general relativity. Here, the gravitational force is derived for a test body held stationary in the exterior geometry of a Kerr black hole.

I. GRAVITATIONAL FORCE

An interesting coordinate system is the case of Boyer-Lindquist coordinates, applicable to the geometry near a rotating, Kerr black hole. The line element in Boyer-Lindquist coordinates can be expressed in many different forms, though a particularly useful form for our purposes here is

$$\begin{aligned} ds^2 =& \frac{\rho^2}{\Sigma^2}\Delta dt^2 - \frac{\rho^2}{\Delta}dr^2 \ldots \\ & - \rho^2 d\theta^2 - \frac{\Sigma^2}{\rho^2}\sin^2\theta\left(d\phi - \frac{2aMr}{\Sigma^2}dt\right)^2 \end{aligned} \tag{1}$$

wherein the functions Δ, ρ^2, and Σ^2 are respectively given as

$$\Delta = r^2 + a^2 - 2Mr \tag{2a}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2b}$$

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta \tag{2c}$$

in which $a = J/M$ is the angular momentum per unit mass, and we have put $GM/c^2 \to M$ for the sake of simplicity.

Let us consider the force on an observer held stationary in Boyer-Lindquist coordinates, with the exception that the observer is dragged along in the $\phi-$direction with an angular velocity $d\phi/dt = 2aMr/\Sigma^2$. The force in general coordinates is given by

$$\mathbf{F} = \frac{1}{2}Eg^{ij}\partial_j\left(\ln(g_{00})\right)\mathbf{g}_i \tag{3}$$

in which $\mathbf{g}_i$ is a basis vector pointing in the $i-$coordinate direction, and E is the total energy of the observer. Referring to Eq. (1), it is straightforward to see that

$$g_{00} = \frac{\rho^2}{\Sigma^2}\Delta. \tag{4}$$

Expressing Eq. (3) in terms of non-zero components, and putting $c = 1$, gives

$$\mathbf{F} = \frac{m}{2g_{00}}\left(g^{11}g_{00,1}\mathbf{g}_1 + g^{22}g_{00,2}\mathbf{g}_2\right). \tag{5}$$

Since working in Boyer-Lindquist coordinates is algebraically involved, let us work on each component of the force separately. Starting with the first term on the right-hand side of Eq. (5), we have

$$g_{00,1} = \frac{\rho^2}{\Sigma^2}\frac{\partial\Delta}{\partial r} + \frac{\Delta}{\Sigma^2}\frac{\partial\rho^2}{\partial r} + \rho^2\Delta\frac{\partial}{\partial r}\left(\frac{1}{\Sigma^2}\right). \tag{6}$$

Carrying out the partial differentiation of Eqs. (2a), (2b), and (2c) gives

$$\frac{\partial\Delta}{\partial r} = 2\left(r - M\right) \tag{7a}$$

$$\frac{\partial\rho^2}{\partial r} = 2r \tag{7b}$$

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{1}{\Sigma^2}\right) = & \frac{-4r}{\Sigma^4}\left(r^2 + a^2\right) + \ldots \\ & + \frac{-2a^2\sin^2\theta}{\Sigma^4}\left(r - M\right). \end{aligned} \tag{7c}$$

Substituting Eqs. (7a), (7b), and (7c) into Eq. (6), and performing a lot of algebraic manipulation, leads to

$$g_{00,1} = -\frac{2}{\Sigma^4}\left[M\rho^2\left(r^4 - a^4\right) + 2a^2r^2M\Delta\sin^2\theta\right]. \tag{8}$$

Upon substituting Eq. (8) into Eq. (5), and noting that $g^{11} = -\Delta/\rho^2$ and $\mathbf{g}_1 = \mathbf{e}_r\rho/\sqrt{\Delta}$, the $r-$component of the force simplifies to

$$\mathbf{F}_r = -m\mathbf{e}_r\left[\frac{M\rho^2\left(r^4 - a^4\right) + 2a^2r^2M\Delta\sin^2\theta}{\rho^3\Sigma^2\sqrt{\Delta}}\right]. \tag{9}$$

Turning now to the second term in Eq. (5), we can start with

$$g_{00,2} = \frac{\Delta}{\Sigma^2}\frac{\partial\rho^2}{\partial\theta} + \rho^2\Delta\frac{\partial}{\partial\theta}\left(\frac{1}{\Sigma^2}\right). \tag{10}$$

Again using Eqs. (2b) and (2c) and carrying out the partial differentiation with respect to θ gives

$$\frac{\partial\rho^2}{\partial\theta} = -2a^2\sin\theta\cos\theta \tag{11a}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\Sigma^2}\right) = \frac{2a^2\Delta}{\Sigma^4}\sin\theta\cos\theta. \tag{11b}$$

Substituting these expressions into Eq. (10) and simplifying a bit leads to

$$g_{00,2} = -\frac{2a^2\Delta}{\Sigma^4}\left(\Sigma^2 - \rho^2\Delta\right)\sin\theta\cos\theta. \tag{12}$$

Substituting Eq. (12) into Eq. (5), and using $g^{22} = -1/\rho^2$ and $\mathbf{g}_2 = \rho\mathbf{e}_\theta$, puts the $\theta-$component of the force in the form

$$\mathbf{F}_\theta = -m\frac{2Mra^2}{\rho^3\Sigma^2}\left(r^2 + a^2\right)\sin\theta\cos\theta\mathbf{e}_\theta. \tag{13}$$

With both components of the force now in hand, we are ready to express the entire gravitational force on the observer. Using Eqs. (9) and (13), and putting $M \to GM/c^2$, the total force on the observer can now be expressed as

$$\begin{aligned}\mathbf{F} = -GMm\mathbf{e}_r&\left[\frac{\rho^2\left(r^4 - a^4\right) + 2a^2r^2\Delta\sin^2\theta}{\rho^3\Sigma^2\sqrt{\Delta}}\right] + \\ &- Gm\mathbf{e}_\theta\frac{2Mra^2}{\rho^3\Sigma^2}\left(r^2 + a^2\right)\sin\theta\cos\theta.\end{aligned} \tag{14}$$

As a final thought, it is worthwhile to verify that Eq. (14) reduces to the expected Schwarzschild solution when the angular momentum, a, is zero. When $a \to 0$, we have $\Delta \to r^2 - 2GM/c^2r^2$, $\rho^2 \to r^2$, and $\Sigma^2 \to r^4$. Using these in Eq. (14), and simplifying a bit, leads directly to

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r\left(1 - \frac{2GM}{c^2r}\right)^{-\frac{1}{2}}. \tag{15}$$

Thus, in the limit $a \to 0$, Eq. (14) reduces to the expected expression of the force on an observer held stationary in Schwarzschild coordinates.

* charles@ridgely.ws