Gravitational Interaction in a Material Medium

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When two gravitating bodies reside in a material medium, Newton's law of universal gravitation must be modified to account for the presence of the medium. A modified expression of Newton's law is known in the literature, but lacks a clear connection with existing gravitational theory. Newton's law in the presence of a homogeneous material medium is herein derived on the basis of classical, Newtonian gravitational theory and by a general relativistic use of Archimedes' principle.

I. INTRODUCTION

Certainly one of the most famous of all equations in physics is Newton's law of universal gravitation, given by

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\mathbf{e}_r, \tag{1}$$

where $\mathbf{F}$ is the vector force acting between two neighboring bodies, m_1 and m_2 are the masses of the bodies, r is the distance between the centers of the bodies, and $\mathbf{e}_r$ is a unit vector pointing in the $r-$coordinate direction. The minus sign indicates that the gravitational force is attractive, and G is the familiar gravitational constant. Virtually every undergraduate text in existence discusses Eq. (1) at length. What doesn't seem to be as widely discussed, if at all, is that when the two bodies reside within a material medium, such as water or air, the force between the bodies is affected by the presence of the medium. Certainly, Eq. (1) is reasonably accurate when the density of the medium is much less than the density of the bodies, such as in the case of air. But when the density of the medium cannot be ignored, Eq. (1) must be modified.

According to Z. Horák,[1] when a material medium is taken into account, Eq. (1) assumes the form

$$\mathbf{F} = -G\frac{m_A m_B}{r^2}\left(1-\frac{\rho_f}{\rho_A}\right)\left(1-\frac{\rho_f}{\rho_B}\right)\mathbf{e}_r, \tag{2}$$

in which ρ_1 and ρ_2 are the mass-densities of the gravitating bodies, and ρ_f is the mass-density of the medium. Equation (2) appears to be little discussed in the literature. One exception is B. Vybíral,[2] who reported several Cavendish-type experiments performed with large metal spheres submerged in water, rather than air, leading to very close agreement with Eq. (2).

Equation (2) predicts that when both ρ_1 and ρ_2 are greater than ρ_f, the force is attractive. When m_1 and m_2 are both zero, however, the force is still attractive. This suggests that if the two bodies are empty cavities in the medium, they attract one another, even though a naive approach to Eq. (1) would seem to suggest otherwise. Moreover, when one body has a density greater than ρ_f and the other body has a density less than ρ_f, the force between the two bodies is repulsive. It is curious that two neighboring cavities attract one another while a cavity and a solid body will repel one another.

In view of some of these predictions, it is difficult to accept Eq. (2), *de facto*, at face value. Indeed, Z. Horak[1] provides a derivation of Eq. (2), but it seems loosely connected with conventional theory. What is missing, is a derivation of Eq. (2) that relies on accepted theory and is digestible at the undergraduate level. In the next section, the gravitational force between two bodies in a material medium is derived based on classical, Newtonian gravitational theory. In Sec. 3, the same problem is approached by use of a generalized form of Archimedes' principle. Both approaches lead directly to expressions equivalent to Eq. (2).

II. CLASSICAL CONSIDERATIONS

A standard way to derive the gravitational force between two bodies is to first derive an expression for the gravitational field potential due to one body and then use the familiar expression $\mathbf{F} = -m\nabla\phi$ to find the force on the second body. In absence of other outside influences, Newton's third law of motion assures that the force on the two bodies is mutual.

Let us consider two spherical bodies, called A and B, having respective mass-densities ρ_A and ρ_B, residing within a homogeneous material medium, having a mass-density ρ_f. The centers of the bodies are separated by a distance r. The gravitational field potential of the first body, say body A, can be computed by use of the familiar expression[3]

$$\phi = -G\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{3}$$

where $\rho(\mathbf{x}')$ is the mass-density of the gravitational source, $\mathbf{x}$ is a vector pointing from the coordinate origin to a point of observation, $\mathbf{x}'$ is a vector pointing from the origin to an elementary portion of the gravitational source, and $|\mathbf{x}-\mathbf{x}'|$ is the magnitude of the vector pointing from the point of observation to the elementary source. Applying Eq. (3) to the region exterior to body A, and carrying out the angular integrations, leaves us with

$$\phi_A = -\frac{4\pi G}{r}\left(\int_0^a \rho_A r'^2 dr' + \int_a^r \rho_f r'^2 dr'\right) + \ldots \\ - 4\pi G \int_r^\infty \rho_f r' dr', \tag{4}$$

in which a is the radius of body A, and the subscript on ϕ_A indicates that the potential is due to body A. Clearly, the right-most integral diverges as a consequence of the extent of the medium. At first sight, this seems a bit confusing. All sources of gravitation ought to be included in Eqs. (3) and (4); but at the same time, we should expect gravitational potential to be finite everywhere.

A way around the divergence of Eq. (4) reveals itself upon considering Gauss' law, expressed for gravitation as[4,5]

$$\oint_S \mathbf{g}\cdot\mathbf{n}\, da = -4\pi G \int_V \rho(\mathbf{x})\, d^3x. \tag{5}$$

The integral on the right-hand side of Eq. (5) expresses the mass contained within an enclosed volume, V, while the left-hand side expresses the gravitational flux passing through the exterior surface, S, of V. The point to notice is that the gravitational flux is proportional to the "net" mass enclosed within the surface.[4] Since the material medium is uniformly distributed everywhere, the medium makes no net contribution to Eq. (4). This suggests that we can normalize Eqs. (3) and (4) so that $\phi \to 0$ in the limit of far fields by essentially subtracting the mass-density of the medium everywhere, including from the bodies A and B. Doing this sets $\rho_f = 0$ outside bodies A and B and puts the mass-densities of bodies A and B in the form

$$\rho_A^* = \rho_A - \rho_f, \tag{6a}$$

$$\rho_B^* = \rho_B - \rho_f. \tag{6b}$$

Returning to Eq. (4) and putting $\rho_f = 0$ eliminates the two right-most integrals, while using Eq. (6a) for the mass-density of body A leads directly to

$$\phi_A = -G\frac{m_A^*}{r}, \tag{7}$$

in which the mass of body A is $m_A^* = \rho_A^* V_A$.

With an expression for the gravitational potential due to body A in hand, we can determine the force on body B by simply taking the gradient of Eq. (7) and then using $\mathbf{F}_B = -m_B^* \nabla\phi_A$. Carrying this out gives the force on body B as

$$\mathbf{F}_B = -G\frac{m_A^* m_B^*}{r^2}\mathbf{e}_r. \tag{8}$$

Equation (8) has the form of Eq. (1) with the exception that m_A^* and m_B^* are expressed in terms of Eqs. (6a) and (6b), respectively. Using Eqs. (6a) and (6b), and rearranging a bit puts Eq. (8) in a form equivalent to Eq. (2):[1,2]

$$\mathbf{F} = -G\frac{m_A m_B}{r^2}\left(1 - \frac{\rho_f}{\rho_A}\right)\left(1 - \frac{\rho_f}{\rho_B}\right)\mathbf{e}_r, \tag{9}$$

in which the subscript has been dropped from $\mathbf{F}_B$ in recognition that the force given by Eq. (9) is mutual between bodies A and B.

III. GENERALIZED ARCHIMEDES' PRINCIPLE

Another way to derive Newton's law in the presence of a material medium is by using Archimedes' principle. Let us again consider the two bodies A and B residing in a uniformly distributed material medium. It is envisioned that each body is suspended by a thin tether so as to maintain a fixed distance r between the centers of the two bodies. Moreover, it is assumed that the two bodies and the medium are in thermodynamic equilibrium, so as to eliminate any phenomena due to differences in temperature.

Equation (2) suggests that each body is buoyed by the medium due to the gravitational field of the other body. So long as the bodies remain stationary, and their respective gravitational fields are time-independent, the vector force on each body is expressible as[6]

$$\mathbf{F} = \frac{mc^2}{2}\frac{g^{ij} g_{00,j}}{g_{00}}\left(1 - \frac{\rho_f}{\rho}\right)\mathbf{e}_i, \tag{10}$$

in which g_{00} and g^{ij} are respective components of the metric tensor $g_{\mu\nu}$ and the inverse metric tensor $g^{\mu\nu}$, c is the speed of light, the comma denotes partial differentiation, and Latin indices are taken over the values $(1, 2, 3)$. The vector $\mathbf{e}_i$ is a basis vector pointing in the $i-$coordinate direction, and ρ is the mass-density of the body experiencing the force. Equation (10) is the apparent weight of each body one would measure upon determining the tension in each tether.

Upon applying Eq. (10) to one body, the metric tensor $g_{\mu\nu}$ should be understood as expressing the space-time curvature due to the combined influence of the other body and the intervening medium. Let us use Eq. (10) to determine the force on body A, due to the presence of body B and the medium. Were the two bodies situated in a vacuum, $g_{\mu\nu}$ due to body B would be expressed in Schwarzschild coordinates. With the medium present, however, there is an additional contribution to $g_{\mu\nu}$. The contribution of the medium can be determined by considering the case in which body B is absent, leaving an empty cavity in the medium. Within the medium, outside the cavity, g_{00} assumes the particularly simple form[7]

$$g_{00} = 1 + \frac{2Gm_f}{rc^2}, \tag{11}$$

where m_f is the mass of the portion of the medium that would otherwise fill the cavity. With body B occupying the cavity, the superposition principle suggests that we may put

$$g_{00} = 1 - \frac{2G}{rc^2}\left(m_B - m_f\right). \tag{12}$$

It is to be noticed that Eq. (12) reduces to the familiar Schwarzschild form when $m_B \neq 0$ and $m_f \to 0$. Substituting Eq. (12) into Eq. (10), and putting $g^{11} = -1$ to limit things to first order terms, leads to

$$\mathbf{F} = -G\frac{m_A m_B}{r^2}\left(1 - \frac{\rho_f}{\rho_A}\right)\left(1 - \frac{\rho_f}{\rho_B}\right)\mathbf{e}_r. \tag{13}$$

Equation (13) is identical to Eqs. (2) and (9),[1,2] as expected. Unlike the approach used in the previous section, however, in this section Eq. (13) was derived solely by applying Archimedes' principle in the form of Eq. (10). Based on this, it would seem that Eq. (13) is ultimately due to Archimedes' principle.

IV. CONCLUSIONS

As discussed in the Introduction, when two gravitating bodies reside in a material medium, Newton's law of universal gravitation must be modified to account for the presence of the medium. Such a modified expression of the force does indeed exist in the literature,[1,2] but appears to lack a clear connection with existing gravitational theory. Herein, Newton's law in the presence of a homogeneous material medium has been derived on the basis of classical, Newtonian gravitational theory and by a general relativistic use of Archimedes' principle.[6] In the former, a direct application of the gravitational field potential to the medium led to divergent results due to the extent of the medium. The solution to this problem hinged on noticing that according to Gauss' law, a uniformly distributed material medium makes no "net" contribution to the gravitational potential. Rather, Gauss' law states that the net gravitational flux emanating from an enclosed surface is proportional to the net mass enclosed within the surface.[4,5] Knowledge of this point justified normalizing the gravitational potential by subtracting the mass-density of the medium everywhere, including from the two gravitating bodies. Carrying this out ensured that the gravitational potential approaches zero in the far-field limit. Working under these conditions led straight to an expression of the gravitational force in agreement with the literature.[1,2]

In Sec. 3, a generalized form of Archimedes' principle was used, in which the buoyancy force on one body was expressed in terms of the metric tensor due to the gravitational field of the other body and the material medium.[6] In order to account for the presence of the medium, the metric tensor in Schwarzschild coordinates was combined with the metric tensor for an empty cavity in the medium.[7] Using the combined metric tensor led directly to the correct expression of the force,[1,2] thus prompting the conclusion that the modified form of Newton's law of universal gravitation ultimately finds its origin in Archimedes' principle.

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