Archimedes' Principle and a Hollow, Fluid-Filled Sphere

Charles T. Ridgely

(Dated: January 18, 2010)

Covariant divergence of the stress-energy-momentum tensor is used to derive the apparent weight of a hollow, fluid-filled sphere which is suspended within a fluid possessing a different density than the fluid inside the sphere.

I. APPARENT WEIGHT OF HOLLOW SPHERE

Let us consider a hollow spherical body of mass, m, and density, ρ, suspended within an exterior fluid of density, ρ_{OUT}. A thin tether supports the sphere within the fluid, and the fluid and sphere are allowed to reach thermodynamic equilibrium. Inside the sphere is a different type of fluid having a mass-density ρ_{IN}. The stress-energy-momentum tensor for the entire system is

$$T^{\mu\nu} = T_m^{\mu\nu} + T_{IN}^{\mu\nu} + T_{OUT}^{\mu\nu} + T_{Ext}^{\mu\nu} \tag{1}$$

where $T_m^{\mu\nu}$ and $T_{IN}^{\mu\nu}$ are the stress-energy-momentum tensors of the sphere and the interior fluid, respectively. The stress-energy-momentum tensor of the exterior fluid is $T_{OUT}^{\mu\nu}$, and $T_{Ext}^{\mu\nu}$ is the stress-energy-momentum tensor of the tether applying an external force on the sphere. At this point, it should be mentioned that $T_m^{\mu\nu}$, $T_{IN}^{\mu\nu}$, and $T_{OUT}^{\mu\nu}$ include all forms of energy and momentum, as well as any internal stress, associated with the sphere, the interior fluid, and the exterior fluid, respectively.

Upon applying covariant divergence of the stress-energy-momentum tensor $T^{\mu\nu}$ and then integrating over the proper volume of the sphere, we arrive at

$$\begin{aligned}\frac{1}{c}\frac{d}{d\tau}\int T^{\mu 0} d^3x + \int \partial_j T^{\mu j} d^3x + \dots \\ - \frac{1}{2} g^{\mu j} g_{00,j} \int T^{00} d^3x = 0\end{aligned} \tag{2}$$

in which τ is proper time, and the metric tensor has been moved outside the integral in the right-most term on the assumption that the gradient of the gravitational potential is roughly uniform across the volume of the sphere. Using Eq. (1) in Eq. (2), and noting that for $T_m^{\mu\nu}$, the second integral must be zero since the sphere doesn't convey energy or momentum across the boundary surface of the sphere, and for $T_{OUT}^{\mu\nu}$ the first and third integrals are zero because there is no exterior fluid within the volume of the sphere, we arrive at

$$\begin{aligned}\frac{1}{c}\frac{d}{d\tau}\int \left(T_m^{\mu 0} + T_{IN}^{\mu 0}\right) d^3x + \dots \\ + \int \partial_j \left(T_{OUT}^{\mu j} + T_{Ext}^{\mu j}\right) d^3x + \dots \\ - \frac{1}{2} g^{\mu j} g_{00,j} \int \left(T_m^{00} + T_{IN}^{00}\right) d^3x = 0.\end{aligned} \tag{3}$$

The first term is easily identified as the time rate of change of the total four-momentum P^μ of the sphere and interior fluid. Expressing the four-force in the general form $f^\mu = dP^\mu/d\tau$ suggests that the the force on the sphere and interior fluid due to changing momentum is

$$f^\mu = \frac{1}{c}\frac{d}{d\tau}\int \left(T_m^{\mu 0} + T_{IN}^{\mu 0}\right) d^3x. \tag{4}$$

Another force on the sphere is the above-mentioned external force acting on the boundary surface of the sphere due to the tether. The force due to the tether is expressible as

$$F^\mu = \int \partial_j T_{Ext}^{\mu j} d^3x. \tag{5}$$

Using Eqs. (4) and (5) puts Eq. (3) in the form

$$\begin{aligned}f^\mu + F^\mu + \int \partial_j T_{OUT}^{\mu j} d^3x + \dots \\ - \frac{1}{2} g^{\mu j} g_{00,j} \int \left(T_m^{00} + T_{IN}^{00}\right) d^3x = 0.\end{aligned} \tag{6}$$

To simplify the third term in Eq. (6), let us return to Eq. (2) and consider the case in which the sphere and interior fluid are absent, and the exterior fluid remains in static equilibrium. In essence, we are considering the forces acting on a volume equal to that of the sphere, filled with the exterior fluid and residing in a larger volume of the same fluid. Using this concept and Eq. (2), and dropping the volume integration, leads to

$$\partial_j T_{OUT}^{\mu j} - \frac{1}{2} g^{\mu j} g_{00,j} T_{OUT}^{00} = 0 \tag{7}$$

in which $dT_{OUT}^{\mu 0}/d\tau = 0$ has been used for the fluid in static equilibrium. Solving for the left-most term in Eq. (7) and then substituting into Eq. (6) leads to

$$\begin{aligned}f^\mu + F^\mu + \frac{1}{2} g^{\mu j} g_{00,j} \int T_{OUT}^{00} d^3x + \dots \\ - \frac{1}{2} g^{\mu j} g_{00,j} \int \left(T_m^{00} + T_{IN}^{00}\right) d^3x = 0.\end{aligned} \tag{8}$$

For the special case when the sphere is held stationary in the fluid, the momentum of the sphere is zero, and thus $f^\mu = 0$ in Eq. (8). Solving for the force on the sphere due to the tether then gives

$$\begin{aligned}F^\mu = \frac{1}{2} g^{\mu j} g_{00,j} \int T_m^{00} d^3x + \dots \\ - \frac{1}{2} g^{\mu j} g_{00,j} \int \left(T_{OUT}^{00} - T_{IN}^{00}\right) d^3x.\end{aligned} \tag{9}$$

Upon noting that in general we may put $T^{00} = \rho(U^0)^2$, and using $U_\mu U^\mu = c^2$, it is straightforward to show that Eq. (9) can be put in the vector form

$$\mathbf{F} = \frac{mc^2}{2}\frac{g^{ij}g_{00,j}}{g_{00}}\left[1 + \frac{m_{IN}}{m}\left(1 - \frac{\rho_{OUT}}{\rho_{IN}}\right)\right]\mathbf{e}_i \qquad (10)$$

in which m_{IN} is the mass of the interior fluid, and $\mathbf{e}_i$ is a unit basis vector pointing in the $i-$coordinate direction. Equation (10) is the apparent weight of the hollow sphere filled with a fluid of density ρ_{IN}, suspended within an exterior fluid of density ρ_{OUT}.